A queue with semi-periodic traffic

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1. Introduction

We analyze the diffusion limit of a discrete time queueing system where sources randomly enter and depart the system. Each source generates periodic traffic while it is in the system. This can model digitized voice traffic for example, where once a connection starts, it generates traffic at constant intervals until it ends. We look at this system to obtain the overflow probability as a function of the the arrival rate, connection duration and period length. In some cases we get exact overflow probabilities, whereas in others we obtain lower and upper bounds. The model allows exploration of the interplay between delay due to random phases and delay due to fluctuations in total load.

2. Summary

Consider a discrete time system, where for integer m, slot m consists of the time interval $[m, m+1)$. Group together N consecutive slots and call them a *frame*. Divide the slots into N equivalence classes, called *phases*. Slots m and l are in the same phase if $m = l + nN$ for some integer n. Therefore, each frame consists of one slot from each of the N phases.

Denote the number of new connections in frame j of phase k by $x_{k,j}$, where $x_{k,j} \sim Poisson(\lambda) \;\; \forall k, j$. Assume that each connection generates 1 packet every N slots, i.e. it sends 1 packet every frame, and always in the same phase. The number of frames the connection remains active is assumed to be geometrically distributed with mean $\frac{1}{1-\alpha}$ for $0 \leq \alpha < 1$, and infinite for $\alpha = 1$. Connections are assumed independent of each other, therefore packet arrivals in different phases are independent of each other.

Denote the number of packets that arrive in slot m by $\tilde{a}_m$, and by $a_{k,j}$ the number of packets that arrive in phase k of frame j, for $k = 0, \ldots, N-1$. This implies that $a_{k,j} = \tilde{a}_{k+Nj}$.

Lemma 1 *For any $k = 0, \ldots, N-1$ and integer j, the distribution of $a_{k,j}$ is Poisson with mean $\rho = \frac{\lambda}{1-\alpha}$. Furthermore, let j_1, j_2 be integers and $k_1, k_2 = 0, \ldots, N-1$, then*

$$E\left[\left(a_{k_1,j_1} - \rho\right)\left(a_{k_2,j_2} - \rho\right)\right] = \rho\alpha^{|j_1 - j_2|}\delta_{k_1,k_2} \tag{1}$$

where $\delta_{k_1,k_2} = 1$ if $k_1 = k_2$ and zero otherwise.

Let $\tilde{A}_m$ denote the cumulative number of arrivals in $[0, m)$, i.e. $\tilde{A}_m = 0$ for $m = 0$, and $\tilde{A}_m = \sum_{l=0}^{m-1} \tilde{a}_l$ for $m \geq 1$. Define on $t \geq 0$

$$X_t^N = \frac{\tilde{A}_{Nt} - \rho\lfloor Nt \rfloor}{\sqrt{\rho N}} \tag{2}$$

Then for any $T > 0$ the process $X_t^N \in D[0, T]$, where $D[0, T]$ is the space of right continuous functions with left limits on the interval $[0, T]$. We consider weak convergence with respect to the Skorohod topology [1] on $D[o, T]$.

Let $\{X_t\}_{t\geq 0}$ be a stationary increment zero mean $a.s.$ continuous Gaussian process with variance:

$$\rho_t := E\left[X_t^2\right] = t + 2\sum_{j=1}^{\infty} \alpha^j (t-j)_+ \quad \forall\, t \geq 0 \tag{3}$$

Proposition 1 *$X_t^N \Rightarrow X_t$ as $N \to \infty$, i.e. X_t^N converges weakly to X_t.*

We provide some discussion on the properties of the limiting process X_t, including its construction from independent Wiener processes. This is done to gain insight into the process.

Denote the buffer size of the discrete time system at time 0_- by Q_0. If $\frac{(1-\rho)}{\sqrt{\rho}}\sqrt{N} \to \theta$ as $N \to \infty$, then by the reversibility of the arrival process and the Continuous Mapping Theorem [2] we get

$$\frac{Q_0}{\sqrt{\rho N}} \Rightarrow \sup_{t\geq 0}\{X_t - \theta t\} \tag{4}$$

We denote the limiting overflow probability as $P_{\alpha,\theta}(\beta)$, where β is assumed to be the limiting buffer size, i.e.

$$P_{\alpha,\theta}(\beta) := P\left[\sup_{t\geq 0}\{X_t - \theta t\} \geq \beta\right] \tag{5}$$

We divide this calculation into three cases depending on the value of α: 1) $\alpha = 0$, 2) $\alpha = 1$, 3) $0 < \alpha < 1$. In the first two cases, the process exhibits particular properties that allow us to obtain exact overflow probabilities, whereas in the third case we present three different lower bounds and an upper bound, and compare them with the empirical process' overflow probability and among themselves. We also discuss the effects of α and θ in these probabilities, as well as in the characteristics of the limiting process X_t in itself.

References

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